

Fatou's Lemma for Weakly Converging Probabilities

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Abstract

Fatou's lemma states under appropriate conditions that the integral of the lower limit of a sequence of functions is not greater than the lower limit of the integrals. This note describes similar inequalities when, instead of a single measure, the functions are integrated with respect to different measures that form a weakly convergent sequence.

1 The Inequality for Nonnegative Functions

Consider a measurable space (S, \mathcal{B}) , where S is a metric space and \mathcal{B} is its Borel σ -field. Let $\mathbb{P}(S)$ be the set of probability measures on $(S, \mathcal{B}(S))$. According to Fatou's lemma, Shiryaev^[Sh] [8], for any $\mu \in \mathbb{P}(S)$ and for any sequence of nonnegative measurable functions f_1, f_2, \dots

$$\int_S \liminf_{n \rightarrow +\infty} f_n(s) \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu(ds). \quad (1.1) \quad \text{eq2}$$

A sequence of probability measures $\{\mu_n\}_{n \geq 1}$ from $\mathbb{P}(S)$ converges weakly to $\mu \in \mathbb{P}(S)$ if for any bounded continuous function f on S

$$\int_S f(s) \mu_n(ds) \rightarrow \int_S f(s) \mu(ds) \quad \text{as } n \rightarrow +\infty. \quad (1.2) \quad \text{eq1}$$

A sequence of probability measures $\{\mu_n\}$ from $\mathbb{P}(S)$ converges setwise to $\mu \in \mathbb{P}(S)$ if (1.2) holds for any bounded measurable function f . If $\{\mu_n\}_{n \geq 1} \subset \mathbb{P}(S)$ converges setwise to $\mu \in \mathbb{P}(S)$, according to Royden^[Ro] [5, p. 231], for any sequence of nonnegative measurable function f_1, f_2, \dots

$$\int_S \liminf_{n \rightarrow +\infty} f_n(s) \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds). \quad (1.3) \quad \text{eq3}$$

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However, this is not true, if μ_1, μ_2, \dots converge weakly to μ .

Indeed, let $S = [0, 1]$, $\mu_n(A) = \mathbf{I}\{1/n \in A\}$, $\mu(A) = \mathbf{I}\{0 \in A\}$ for $A \in \mathcal{B}([0, 1])$, and $f(s) = f_n(s) = \mathbf{I}\{s = 0\}$ for $n = 1, 2, \dots$ and $s \in [0, 1]$. Then $\int_S f(s)\mu(ds) = 1$, $\int_S f(s)\mu_n(ds) = 0$, and (1.3) does not hold.

Theorem 1.1 presents Fautou's lemma for weakly converging measures μ_n and nonnegative functions f_n . This fact is useful fact for the analysis of Markov decision processes and stochastic games. Serfozo [7, Lemma 3.2] establishes inequality (1.4) for a vaguely convergent sequence of measures on a locally compact metric space S and for nonnegative functions f_n . In its current form, Theorem 1.1 is formulated in Schäl [6, Lemma 2.3(ii)] without proof, in Jaskiewicz and Nowak [4, Lemma 3.2] with short explanations on how the proof from Serfozo [7, Lemma 3.2] can be adapted to weak convergence on metric spaces, and in Feinberg, Kasyanov, and Zadoianchuk [3, Lemma 4] with a proof. To make this note logically complete, we provide the proof of Theorem 1.1 in Section 3 below. The provided proof is shorter and simpler than the proof in [3]. Theorem 4.3 below extends Theorem 1.1 to functions f_n that can be unbounded from below. Lemma 3.3 in Jaskiewicz and Nowak [4] is a particular version of such a result developed for particular applications in that paper. Let $\overline{\mathbb{R}} = [-\infty, +\infty]$.

lemma2

Theorem 1.1. *Let S be an arbitrary metric space, $\{\mu_n\}_{n \geq 1} \subset \mathbb{P}(S)$ converge weakly to $\mu \in \mathbb{P}(S)$, and $\{f_n\}_{n \geq 1}$ be a sequence of measurable nonnegative $\overline{\mathbb{R}}$ -valued functions on S . Then*

$$\int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds). \quad (1.4) \quad \text{eq3.1}$$

We remark that, if $f_n(s) = f(s)$, $n = 1, 2, \dots$, and the function f is nonnegative and lower semicontinuous then $\liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') = f(s)$ and Theorem 1.1 implies that

$$\int_S f(s) \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S f(s) \mu_n(ds), \quad (1.5) \quad \text{eq:J0}$$

if μ_n converges weakly to μ ; see Billingsley [1, problem 7, Chapter 1, §2], where this fact is stated for a bounded lower semicontinuous f .

Further, for any $\overline{\mathbb{R}}$ -valued function u on S we denote

$$\underline{u}(s) = \liminf_{s' \rightarrow s} u(s'), \quad \overline{u}(s) = \limsup_{s' \rightarrow s} u(s'), \quad s \in S.$$

Theorem 4.3 below provides the extended version of Theorem 1.1 for unbounded below functions.

2 Proof of Theorem 1.1

Proof. First, we prove the lemma for uniformly bounded above functions f_n . Let $f_n(s) \leq K < +\infty$ for all $n = 1, 2, \dots$ and all $s \in S$. For $n = 1, 2, \dots$ and $s \in S$, define $F_n(s) = \inf_{m \geq n} f_m(s)$.

The functions $\underline{F}_n : S \rightarrow [0, +\infty]$, $n = 1, 2, \dots$, are lower semi-continuous; see, Berberian [\[2, Lemma 5.13.4\]](#). In addition, for $s \in S$

$$\underline{F}_n(s) \uparrow \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') \quad \text{as } n \rightarrow +\infty. \quad (2.1) \quad \text{eq:kb0}$$

By the monotone convergence theorem,

$$\int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') \mu(ds) = \lim_{n \rightarrow +\infty} \int_S \underline{F}_n(s) \mu(ds). \quad (2.2) \quad \text{eq:J1}$$

Since the function \underline{F}_n , $n = 1, 2, \dots$, is lower semi-continuous on S and bounded below and μ_m converges weakly to μ as $m \rightarrow +\infty$, then formula [\(1.5\)](#) provides

$$\int_S \underline{F}_n(s) \mu(ds) \leq \liminf_{m \rightarrow +\infty} \int_S \underline{F}_n(s) \mu_m(ds), \quad n = 1, 2, \dots \quad (2.3) \quad \text{eq:J2}$$

Because of \underline{F}_n is monotonically nondecreasing by $n = 1, 2, \dots$, then

$$\liminf_{m \rightarrow +\infty} \int_S \underline{F}_n(s) \mu_m(ds) \leq \liminf_{m \rightarrow +\infty} \int_S \underline{F}_m(s) \mu_m(ds), \quad n = 1, 2, \dots \quad (2.4) \quad \text{eq:J3}$$

Formulas [\(2.2\)](#)–[\(2.4\)](#) provide necessary inequality [\(1.4\)](#).

Thus Theorem [1.1](#) is proved for uniformly bounded functions f_n . Consider a sequence $\{f_n\}_{n \geq 1}$ of measurable nonnegative $\overline{\mathbb{R}}$ -valued functions on S . For $\lambda > 0$ set $f_n^\lambda(s) := \min\{f_n(s), \lambda\}$, $s \in S$, $n = 1, 2, \dots$. Since the functions f_n^λ are uniformly bounded above,

$$\int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n^\lambda(s') \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S f_n^\lambda(s) \mu_n(ds) \leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds).$$

Then, using Fatou's lemma,

$$\int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') \mu(ds) \leq \liminf_{\lambda \rightarrow +\infty} \int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n^\lambda(s') \mu(ds).$$

□

3 A Counterexample for Functions Unbounded Below

s3

A suitable assumption concerning the negative parts of the sequence f_1, f_2, \dots of functions is necessary for Fatou's lemma for weakly converging probabilities as well as for setwise converging probabilities, as the following example shows.

exa1

Example 3.1. The sequence of probability measures $\{\mu_n\}_{n \geq 1}$ converges setwise (and therefore converges weakly) to a probability measure μ from $\mathbb{P}(S)$, real function $f : S \rightarrow \mathbb{R}$ is continuous,

$$\int |f(s)| \mu(ds), \int |f(s)| \mu_n(ds) < +\infty, \quad n \geq 1,$$

and

$$\int f(s)\mu(ds) > \lim_{n \rightarrow +\infty} \int f(s)\mu_n(ds).$$

Let S denote the semiinterval $(0, 1]$ with the Borel σ -field $\mathcal{B}(S)$. For every natural number n define probability measure

$$\mu_n(A) = \sqrt{n}\lambda\left(A \cap \left[\frac{1}{2n}, \frac{1}{n}\right]\right) + \left(2 - \frac{1}{\sqrt{n}}\right)\lambda\left(A \cap \left[\frac{1}{2}, 1\right]\right), \quad A \in \mathcal{B}(S),$$

where λ is the Lebesgue measure on $(0, 1]$. Define also continuous on S real function $f(s) = -s^{-1}$. The sequence of probability measures $\{\mu_n\}_{n \geq 1}$ converges setwise (and therefore converges weakly) to the probability measure μ from $\mathbb{P}(S)$, where $\mu(A) = 2\lambda\left(A \cap \left[\frac{1}{2}, 1\right]\right)$, $A \in \mathcal{B}(S)$, and

$$\int f(s)\mu(ds) = -2\ln(2), \quad \int f(s)\mu_n(ds) = -\ln(2) \left(\sqrt{n} + 2 - \frac{1}{\sqrt{n}}\right), \quad n \geq 1.$$

Thus

$$\int f(s)\mu(ds) > \lim_{n \rightarrow +\infty} \int f(s)\mu_n(ds) = -\infty.$$

Remark 3.2. If we set $f(s) = s^{-1}$ for $s \in (0, 1]$, $n \geq 1$, in example [exal](#), then inequalities [\(I.3\)](#) and [\(I.4\)](#) are strict.

4 Extensions and Variations

In the rest of this paper, we deal with integrals of functions that can take negative values. An integral $\int_S f(s)\mu(ds)$ of a measurable $\overline{\mathbb{R}}$ -valued function f on S with respect to a probability measure $\mu \in \mathbb{P}(S)$ is defined if

$$\min\left\{\int_S f^+(s)\mu(ds), \int_S f^-(s)\mu(ds)\right\} < +\infty, \quad (4.1) \quad \text{e:condint}$$

where $f^+(s) = \max\{f(s), 0\}$, $f^-(s) = -\min\{f(s), 0\}$, $s \in S$. If [\(4.1\)](#) holds then the integral is defined as

$$\int_S f(s)\mu(ds) = \int_S f^+(s)\mu(ds) - \int_S f^-(s)\mu(ds).$$

All the integrals in the assumptions of the following theorems and corollary are assumed to be defined. For example, by writing [\(4.2\)](#) in Theorem [4.1](#), we assume that the integrals are defined for the functions $g_n(s)$, $n \geq 1$, and $\limsup_{n \rightarrow +\infty} g_n(s)$.

The following statement is a generalization of [\(II.3\)](#) to functions that can take negative values.

[teor2](#)

Theorem 4.1. Let $\{\mu_n\}_{n \geq 1} \subset \mathbb{P}(S)$ converge setwise to $\mu \in \mathbb{P}(S)$ and let $\{f_n\}_{n \geq 1}$ be a sequence of measurable $\overline{\mathbb{R}}$ -valued functions defined on $(S, \mathcal{B}(S))$. Then inequality [\(I.3\)](#) holds, if all the integrals in [\(I.3\)](#) are defined and there exists a sequence of measurable \mathbb{R} -valued functions $\{g_n\}_{n \geq 1}$ on S such that $f_n(s) \geq g_n(s)$, for all $n \geq 1$ and for all $s \in S$, and

$$-\infty < \int_S \limsup_{n \rightarrow +\infty} g_n(s)\mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S g_n(s)\mu_n(ds). \quad (4.2) \quad \text{eq:sw1}$$

Proof. If at least one of the inequalities

$$\liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds) < +\infty, \quad -\infty < \int_S \liminf_{n \rightarrow +\infty} f_n(s) \mu(ds) \quad (4.3) \quad \text{eq:fl}$$

is violated then inequality (I.3) holds. So, we assume (4.3). The left inequality in (4.3) implies

$$\liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds) < +\infty. \quad (4.4) \quad \text{eq:flg}$$

Let us apply Fatou's lemma for setwise converging probabilities (see (I.3)) to the sequence $\{f_n - g_n\}_{n \geq 1}$ of nonnegative $\overline{\mathbb{R}}$ -valued measurable functions on S . Then

$$\int_S \liminf_{n \rightarrow +\infty} (f_n(s) - g_n(s)) \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S (f_n(s) - g_n(s)) \mu_n(ds). \quad (4.5) \quad \text{eq:fl1}$$

Inequalities (4.2) and (4.4) imply

$$-\infty < \int_S \limsup_{n \rightarrow +\infty} g_n(s) \mu(ds) < +\infty. \quad (4.6) \quad \text{eq1gn}$$

In view of (4.6) and the right inequality in (4.3),

$$\liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu(ds) - \limsup_{n \rightarrow +\infty} \int_S g_n(s) \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S (f_n(s) - g_n(s)) \mu(ds) \quad \mu(ds)\text{-a.s.}, \quad (4.7) \quad \text{eq:fl1EF1}$$

and

$$\int_S \liminf_{n \rightarrow +\infty} f_n(s) \mu(ds) - \int_S \limsup_{n \rightarrow +\infty} g_n(s) \mu(ds) \leq \int_S \liminf_{n \rightarrow +\infty} (f_n(s) - g_n(s)) \mu(ds). \quad (4.8) \quad \text{eq:fl1EF}$$

The following inequalities and (4.6) imply (I.3) since

$$\begin{aligned} \int_S \liminf_{n \rightarrow +\infty} f_n(s) \mu(ds) - \int_S \limsup_{n \rightarrow +\infty} g_n(s) \mu(ds) &\leq \liminf_{n \rightarrow +\infty} \int_S (f_n(s) - g_n(s)) \mu_n(ds) \\ &\leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds) - \liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds) \\ &\leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds) - \int_S \limsup_{n \rightarrow +\infty} g_n(s) \mu(ds), \end{aligned}$$

where the first inequality follows from (4.8) and (4.5), the second one holds since $-\infty < \liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds) < +\infty$ in view of (4.2), (4.3), and $g_n \leq f_n$, and the last inequality holds because of (4.2) and (4.6). \square

Remark 4.2. The second inequality in (4.2) coincides with (I.3), when $f_n = g_n = g$, $n = 1, 2, \dots$

The following theorem extends Theorem I.1 to functions that can take negative values.

teor3

Theorem 4.3. Let S be an arbitrary metric space, $\{\mu_n\}_{n \geq 1} \subset \mathbb{P}(S)$ converge weakly to $\mu \in \mathbb{P}(S)$, and $\{f_n\}_{n \geq 1}$ be a sequence of measurable $\overline{\mathbb{R}}$ -valued functions on S . Then inequality (I.4) holds, if all the integrals in (I.4) are defined and there exists a sequence of measurable \mathbb{R} -valued functions $\{g_n\}_{n \geq 1}$ on S such that $f_n(s) \geq g_n(s)$, for all $n \geq 1$ and for all $s \in S$, and

$$-\infty < \int_S \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds). \quad (4.9) \quad \text{eq:sw2}$$

Proof. If at least one of the inequalities

$$\liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds) < +\infty, \quad -\infty < \int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') \mu(ds). \quad (4.10) \quad \text{eq:f11}$$

is violated then inequality (1.4) holds. So, we assume (4.10). The left inequality in (4.10) implies

$$\liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds) < +\infty. \quad (4.11) \quad \text{eq:flg1}$$

Inequalities (4.9) and (4.11) imply that

$$-\infty < \int_S \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \mu(ds) < +\infty. \quad (4.12) \quad \text{eq:lg1}$$

In view of (4.12) and the right inequality in (4.10),

$$\liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') - \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \leq \liminf_{n \rightarrow +\infty, s' \rightarrow s} [f_n(s') - g_n(s')] \quad \mu(ds)\text{-a.s.}, \quad (4.13) \quad \text{eq:k1}$$

and

$$\int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') \mu(ds) - \int_S \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \mu(ds) \leq \int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} h_n(s') \mu(ds), \quad (4.14) \quad \text{eq:un1}$$

where $h_n(s) = f_n(s) - g_n(s)$, $s \in S$, $n = 1, 2, \dots$.

Let us apply Fatou's lemma for weak converging probabilities (see Theorem 1.1) to the sequence $\{h_n\}_{n \geq 1}$ of nonnegative \mathbb{R} -valued measurable functions on S . Then

$$\int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} h_n(s') \mu(ds) \leq \liminf_{n \rightarrow +\infty} \int_S h_n(s) \mu_n(ds). \quad (4.15) \quad \text{eq:un1a}$$

Since $-\infty < \liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds) < +\infty$ in view of (4.9), (4.11), and $g_n \leq f_n$, we have

$$\liminf_{n \rightarrow +\infty} \int_S h_n(s) \mu_n(ds) \leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds) - \liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds). \quad (4.16) \quad \text{eq:un2}$$

The following inequalities (4.14)–(4.16) and (4.12) imply (1.4) since

$$\begin{aligned} & \int_S \liminf_{n \rightarrow +\infty, s' \rightarrow s} f_n(s') \mu(ds) - \int_S \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \mu(ds) \\ & \leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds) - \liminf_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds) \\ & \leq \liminf_{n \rightarrow +\infty} \int_S f_n(s) \mu_n(ds) - \int_S \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \mu(ds), \end{aligned}$$

where the first inequality follows from (4.14) and (4.16), and the second inequality holds because of (4.9) and (4.12). \square

rem:bb

Remark 4.4. Observe that, if the functions $f_n(s) \geq K > -\infty$ for any $s \in S$ and $n = 1, 2, \dots$, in Theorem 4.3, then $g_n(s) = K$ for any $s \in S$ and $n = 1, 2, \dots$, and assumption (4.9) holds. This fact also follows from Theorem 1.1.

Remark 4.5. Example ^{exa1}5.1 demonstrates that assumptions ^{eq:sw1}(4.2) and ^{eq:sw2}(4.9) are essential for Theorems ^{teor2}4.1 and ^{teor3}4.3 respectively.

Remark 4.6. Theorem ^{lemma2}4.1 yields that, for uniformly bounded above functions $\{g_n\}_{n \geq 1}$, assumption ^{eq:sw2}(4.9) in Theorem ^{teor3}4.3 is equivalent to

$$\int_S \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \mu(ds) = \lim_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds) > -\infty. \quad (4.17) \quad \boxed{\text{eq:E}}$$

Indeed, applying Fatou's lemma for uniformly bounded below functions $\{-g_n\}_{n \geq 1}$ (see Remark ^{rem:bb}4.4) we obtain the inequality

$$\int_S \limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') \mu(ds) \geq \limsup_{n \rightarrow +\infty} \int_S g_n(s) \mu_n(ds),$$

that together with assumption ^{eq:sw2}(4.9) imply ^{eq:E}(4.17).

cor **Corollary 4.7.** Let S be an arbitrary metric space, $\{\mu_n\}_{n \geq 1} \subset \mathbb{P}(S)$ converge weakly to $\mu \in \mathbb{P}(S)$, and $\{f_n\}_{n \geq 1}$ be a sequence of measurable $\overline{\mathbb{R}}$ -valued functions on S . Then inequality ^{eq3.1}(4.4) holds, if there exists a bounded above measurable \mathbb{R} -valued function g on S such that $f_n(s) \geq g(s)$ for all $n \geq 1$ and $s \in S$, and

$$-\infty < \int_S \overline{g}(s) \mu(ds) = \lim_{n \rightarrow \infty} \int_S g(s) \mu_n(ds). \quad (4.18) \quad \boxed{\text{eq:sw2aaa}}$$

Remark 4.8. If function g from Corollary ^{cor}4.7 is upper semi-continuous (in particular, continuous), then Assumption ^{eq:sw2aaa}(4.18) has the following form:

$$-\infty < \int_S g(s) \mu(ds) = \lim_{n \rightarrow \infty} \int_S g(s) \mu_n(ds).$$

In the following example functions $\{f_n\}_{n \geq 1}$ are unbounded below and the assumptions of Theorem ^{teor3}4.3 are satisfied.

Example 4.9. Let $S = \mathbb{Q}$ be the set of rational numbers with the metric $\rho(s_1, s_2) = |s_1 - s_2|$, $s_1, s_2 \in S$. We number the elements of $S = \{x_i\}_{i \geq 1}$ and set $f_n = g_n = -n \mathbf{I}\{s \in D_n\}$, where $D_n = \{x_1, x_2, \dots, x_n\}$, $n = 1, 2, \dots$. Note that $\limsup_{n \rightarrow +\infty, s' \rightarrow s} g_n(s') = 0$ for any $s \in S$.

We consider an increasing sequence of natural numbers $\{k_n\}_{n \geq 1} \subset \mathbb{N}$ such that $\frac{k_n}{k_n+1} \notin D_n$, $n = 1, 2, \dots$. Let us set

$$\mu_n(B) = \mathbf{I} \left\{ \frac{k_n}{k_n+1} \in B \right\}, \quad \mu(B) = \mathbf{I} \{1 \in B\}, \quad B \in \mathcal{B}(S), \quad n = 1, 2, \dots$$

The sequence of probability measures $\{\mu_n\}_{n \geq 1} \subset \mathbb{P}(S)$ converges weakly to $\mu \in \mathbb{P}(S)$. Moreover, assumption ^{eq:sw2}(4.9) holds. Therefore, Theorem ^{teor3}4.3 implies ^{eq3.1}(4.4).

We remark that $g(s) = -\infty$ for all $s \in S$ for any function g such that $g(s) \leq f_n(s)$ for all $n = 1, 2, \dots$ and for all $s \in S$. Thus, $\overline{g}(s) = -\infty$ for all $s \in S$, assumption ^{eq:sw2aaa}(4.18) does not hold, and Corollary ^{cor}4.7 is not applicable to this example.

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